

COMMUTATIVE POST-LIE ALGEBRA STRUCTURES ON LIE ALGEBRAS

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ABSTRACT. We show that any CPA-structure (commutative post-Lie algebra structure) on a perfect Lie algebra is trivial. Furthermore we give a general decomposition of inner CPA-structures, and classify all CPA-structures on parabolic subalgebras of simple Lie algebras.

1. INTRODUCTION

Post-Lie algebras have been introduced by Valette in connection with the homology of partition posets and the study of Koszul operads [18]. Loday [14] studied pre-Lie algebras and post-Lie algebras within the context of algebraic operad triples. We rediscovered post-Lie algebras as a natural common generalization of pre-Lie algebras [11, 12, 17, 2, 3, 4] and LR-algebras [6, 7] in the geometric context of nil-affine actions of Lie groups. We then studied post-Lie algebra structures in general, motivated by the importance of pre-Lie algebras in geometry, and in connection with generalized Lie algebra derivations [8, 9, 10]. In particular, the existence question of post-Lie algebra structures on a given pair of Lie algebras turned out to be very interesting and quite challenging. But even if existence is clear the question remains how many structures are possible. In [10] we introduced a special class of post-Lie algebra structures, namely *commutative* ones. We conjectured that any commutative post-Lie algebra structure, in short CPA-structure, on a complex, perfect Lie algebra is *trivial*. For several special cases we already proved the conjecture in [10], but the general case remained open. One main result of this article here is a full proof of this conjecture, see Theorem 3.3. Furthermore we also study inner CPA-structures and give a classification of CPA-structures on parabolic subalgebras of semisimple Lie algebras.

In section 2 we study ideals of CPA-structures, non-degenerate and inner CPA-structures. In particular we show that any CPA-structure on a complete Lie algebra is inner. We give a general decomposition of inner CPA-structures, see Theorem 2.14. This implies, among other things, that any Lie algebra \mathfrak{g} admitting a non-degenerate inner CPA-structure is metabelian, i.e., satisfies $[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] = 0$.

In section 3 we prove the above conjecture and generalize the result to perfect subalgebras of arbitrary Lie algebras in Theorem 3.4. This also implies that any Lie algebra admitting a non-degenerate CPA-product is solvable. Conversely we show that any non-trivial solvable Lie algebra admits a non-trivial CPA-product.

In section 4 we classify all CPA-structures on parabolic subalgebras of simple Lie algebras in Theorem 4.8. We obtain an explicit description of these products for standard Borel subalgebras of simple Lie algebras.

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2. PRELIMINARIES

Let K always denote a field of characteristic zero. Post-Lie algebra structures on pairs of Lie algebras $(\mathfrak{g}, \mathfrak{n})$ over K are defined as follows [8]:

Definition 2.1. Let $\mathfrak{g} = (V, [,])$ and $\mathfrak{n} = (V, \{, \})$ be two Lie brackets on a vector space V over K . A *post-Lie algebra structure* on the pair $(\mathfrak{g}, \mathfrak{n})$ is a K -bilinear product $x \cdot y$ satisfying the identities:

$$\begin{aligned} (1) \quad & x \cdot y - y \cdot x = [x, y] - \{x, y\} \\ (2) \quad & [x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z) \\ (3) \quad & x \cdot \{y, z\} = \{x \cdot y, z\} + \{y, x \cdot z\} \end{aligned}$$

for all $x, y, z \in V$.

Define by $L(x)(y) = x \cdot y$ and $R(x)(y) = y \cdot x$ the left respectively right multiplication operators of the algebra $A = (V, \cdot)$. By (3), all $L(x)$ are derivations of the Lie algebra $(V, \{, \})$. Moreover, by (2), the left multiplication

$$L: \mathfrak{g} \rightarrow \text{Der}(\mathfrak{n}) \subseteq \text{End}(V), \quad x \mapsto L(x)$$

is a linear representation of \mathfrak{g} . A particular case of a post-Lie algebra structure arises if the algebra $A = (V, \cdot)$ is *commutative*, i.e., if $x \cdot y = y \cdot x$ is satisfied for all $x, y \in V$. Then the two Lie brackets $[x, y] = \{x, y\}$ coincide, and we obtain a commutative algebra structure on V associated with only one Lie algebra [10].

Definition 2.2. A *commutative post-Lie algebra structure*, or *CPA-structure* on a Lie algebra \mathfrak{g} is a K -bilinear product $x \cdot y$ satisfying the identities:

$$\begin{aligned} (4) \quad & x \cdot y = y \cdot x \\ (5) \quad & [x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z) \\ (6) \quad & x \cdot [y, z] = [x \cdot y, z] + [y, x \cdot z] \end{aligned}$$

for all $x, y, z \in V$. The associated algebra $A = (V, \cdot)$ is called a CPA.

There is always the *trivial* CPA-structure on \mathfrak{g} , given by $x \cdot y = 0$ for all $x, y \in \mathfrak{g}$. However, in general it is not obvious whether or not a given Lie algebra admits a non-trivial CPA-structure. For abelian Lie algebras, CPA-structures correspond to commutative associative algebras:

Example 2.3. Suppose that (A, \cdot) is a CPA-structure on an abelian Lie algebra \mathfrak{g} . Then A is commutative and associative.

Indeed, using (4), (5) and $[x, y] = 0$ we have

$$x \cdot (z \cdot y) = x \cdot (y \cdot z) = y \cdot (x \cdot z) = (x \cdot z) \cdot y$$

for all $x, y, z \in \mathfrak{g}$.

It is easy to see that there are examples only admitting trivial CPA-structures:

Example 2.4. Every CPA-structure on $\mathfrak{sl}_2(K)$ is trivial.

This follows from a direct computation, but also holds true more generally for every semisimple Lie algebra, see Proposition 3.1. One main aim of this paper is to show that this is even true for all *perfect* Lie algebras, see Theorem 3.3.

Definition 2.5. A CPA-structure (A, \cdot) on \mathfrak{g} is called *nondegenerate* if the annihilator

$$\text{Ann}_A = \ker(L) = \{x \in \mathfrak{g} \mid L(x) = 0\}$$

is trivial.

Note that Ann_A is an ideal of the CPA as well as an ideal of the Lie algebra. Here a subspace I of V is an algebra ideal if $A \cdot I \subseteq I$, and a Lie algebra ideal if $[\mathfrak{g}, I] \subseteq I$. An *ideal* is defined to be an ideal for both A and \mathfrak{g} . Let $[x_1, \dots, x_n] := [x_1, [x_2, [x_3, \dots, x_n]]] \cdots$ and $I^{[n]} := [I, [I, [I, \dots]]] \cdots$ for an ideal I .

Proposition 2.6. *Suppose that (A, \cdot) is a CPA-structure on \mathfrak{g} . Then there exists an ideal I_∞ such that*

- (1) $I_\infty^{[k]} \subseteq \text{Ann}_A \subseteq I_\infty$ for all k large enough.
- (2) *The CPA-structure on \mathfrak{g}/I_∞ is nondegenerate.*

Proof. Define an ascending chain of ideals I_n by $I_0 = 0$ and $I_n = \{x \in A \mid x \cdot A \subseteq I_{n-1}\}$ for $n \geq 1$. We have $I_1 = \text{Ann}_L$ and each I_n is indeed an ideal because of $I_n \cdot A \subseteq I_{n-1} \subseteq I_n$, and

$$\begin{aligned} [I_n, \mathfrak{g}] \cdot A &\subseteq I_n \cdot (\mathfrak{g} \cdot A) + \mathfrak{g} \cdot (I_n \cdot A) \\ &\subseteq I_n \cdot A + \mathfrak{g} \cdot I_{n-1} \\ &\subseteq I_{n-1}. \end{aligned}$$

So for $x \in [I_n, \mathfrak{g}]$ and $a \in A$ we have $x \cdot a \in I_{n-1}$, hence $x \in I_n$. Since \mathfrak{g} is finite-dimensional, this chain stabilizes, i.e., there exists a minimal k such that $I_k = I_\ell$ for all $\ell \geq k$. Then define $I_\infty := I_k$. By construction we have $A \cdot I_1 = 0$, $A \cdot (A \cdot I_2) \subseteq A \cdot I_1 = 0$, etc., so that right-associative products in I_∞ of length at least $k+1$ vanish. Using (5) we have

$$[x_1, \dots, x_{n-1}, x_n] \cdot z = [x_1, \dots, x_{n-1}] \cdot (x_n \cdot z) - x_n \cdot ([x_1, \dots, x_{n-1}] \cdot z)$$

for all $x, y, z \in V$. By induction we see that the elements $[x_1, \dots, x_n] \cdot z$ are spanned by the right-associative elements $x_{\pi(1)} \cdot x_{\pi(2)} \cdots x_{\pi(n)} \cdot z$, where π runs over all permutations in S_n . This yields $I_\infty^{[k+1]} \cdot \mathfrak{g} = 0$, and hence $I_\infty^{[k+1]} \subseteq \text{Ann}_A$. We also have $\text{Ann}_A = I_1 \subseteq I_\infty$. Furthermore $x \cdot \mathfrak{g} \subseteq I_\infty$ implies $x \in I_\infty$, so that the induced CPA-structure on \mathfrak{g}/I_∞ is nondegenerate. Note that I_∞ is in fact the minimal ideal with this property. \square

Definition 2.7. A CPA-structure on \mathfrak{g} is called *weakly inner*, if there is a $\varphi \in \text{End}(V)$ such that the algebra product is given by

$$x \cdot y = [\varphi(x), y].$$

It is called *inner*, if in addition φ is a Lie algebra homomorphism, i.e., $\varphi \in \text{End}(\mathfrak{g})$.

In terms of operators this means that we have $L(x) = \text{ad}(\varphi(x))$ for all $x \in V$. We have $\ker(\varphi) \subseteq \ker(L)$ with equality for $Z(\mathfrak{g}) = 0$.

Lemma 2.8. *Let \mathfrak{g} be a Lie algebra with trivial center. Then any weakly inner CPA-structure on \mathfrak{g} is inner.*

Proof. A product $x \cdot y = [\varphi(x), y]$ with some $\varphi \in \text{End}(V)$ defines a CPA-structure on \mathfrak{g} , if and only if

$$\begin{aligned} [\varphi(x), y] &= [\varphi(y), x] \\ [[\varphi(x), \varphi(y)], z] &= [\varphi([x, y]), z] \end{aligned}$$

for all $x, y, z \in \mathfrak{g}$. In case that $Z(\mathfrak{g}) = 0$ the last condition says that φ is a Lie algebra homomorphism. \square

Corollary 2.9. *Let \mathfrak{g} be a complete Lie algebra. Then any CPA-structure on \mathfrak{g} is inner.*

Proof. By definition we have $\text{Der}(\mathfrak{g}) = \text{ad}(\mathfrak{g})$ and $Z(\mathfrak{g}) = 0$. Hence $L(x) \in \text{Der}(\mathfrak{g})$ implies that $L(x) = \text{ad}(\varphi(x))$ for some $\varphi \in \text{End}(\mathfrak{g})$. \square

In general not all CPA-structures on a Lie algebra are inner or weakly inner. This is trivially the case for abelian Lie algebras, which do admit nonzero CPA-structures, which cannot be weakly inner. The Heisenberg Lie algebra $\mathfrak{h}_1 = \langle e_1, e_2, e_3 \mid [e_1, e_2] = e_3 \rangle$ admits a family $A(\mu)$ of CPA-structures given by $e_1 \cdot e_1 = e_2$, $e_1 \cdot e_2 = e_2 \cdot e_1 = \mu e_3$ for $\mu \in K$, see Proposition 6.3 in [10]:

Example 2.10. *The CPA-product $A(\mu)$ on the Heisenberg Lie algebra \mathfrak{h}_1 is not weakly inner.*

Indeed, all $\text{ad}(\varphi(x))$ map \mathfrak{h}_1 into its center, whereas $L(e_1)$ does not. Hence $L(x) = \text{ad}(\varphi(x))$ cannot hold for all $x \in \mathfrak{h}_1$.

Lemma 2.11. *Suppose that (A, \cdot) is an inner CPA-structure on \mathfrak{g} . Then the ascending chain of ideals I_n is invariant under φ , and all Lie algebra ideals of \mathfrak{g} are ideals of A . Conversely, if the structure is nondegenerate, all ideals of A are Lie algebra ideals.*

Proof. Let I be a Lie algebra ideal. Then $\mathfrak{g} \cdot I = [\varphi(\mathfrak{g}), I] \subseteq [\mathfrak{g}, I] \subseteq I$. Conversely, let I be an algebra ideal and (A, \cdot) be nondegenerate, given by $x \cdot y = [\varphi(x), y]$ with φ being invertible. Then $\varphi(\mathfrak{g}) = \mathfrak{g}$, so that

$$[\mathfrak{g}, I] = [\varphi(\mathfrak{g}), I] = \mathfrak{g} \cdot I \subseteq I.$$

The ideals I_n were defined by $I_0 = 0$ and $I_n = \{x \in A \mid x \cdot A \subseteq I_{n-1}\}$ for $n \geq 1$. Clearly $\varphi(I_0) = I_0$. Using induction we obtain

$$\begin{aligned} \mathfrak{g} \cdot \varphi(I_n) &= [\varphi(\mathfrak{g}), \varphi(I_n)] \\ &= \varphi([\mathfrak{g}, I_n]) \\ &\subseteq \varphi(I_{n-1}) \subseteq I_{n-1}. \end{aligned}$$

Hence $\varphi(I_n) \subseteq I_n$ for all n . \square

Lemma 2.12. *Suppose that $x \cdot y = [\varphi(x), y]$ is an inner CPA-structure on a complex Lie algebra \mathfrak{g} , and let $\mathfrak{g} = \bigoplus_{\alpha} \mathfrak{g}_{\alpha}$ be the generalized eigenspace decomposition of \mathfrak{g} with respect to φ . Then we have*

$$\begin{aligned} [\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] &\subseteq \mathfrak{g}_{\alpha\beta}, \\ [\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] &\neq 0 \text{ implies } \alpha + \beta = 0. \end{aligned}$$

Proof. The first statement is well-known, so that we only need to prove the second one. Using $[\varphi(x), y] = -[x, \varphi(y)]$ we obtain

$$\varphi([x, y]) = [\varphi(x), \varphi(y)] = -[\varphi^2(x), y].$$

By induction on $k \geq 0$ this yields

$$(\varphi + \gamma \text{id})^k([x, y]) = (-1)^k \cdot [(\varphi^2 - \gamma \text{id})^k(x), y].$$

The RHS vanishes for $\gamma := \alpha^2$ and k large enough, since if φ has a generalized eigenvector x with generalized eigenvalue α , then φ^2 has generalized eigenvalue α^2 for x . This yields $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{-\alpha^2}$, and similarly $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{-\beta^2}$, hence

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{-\alpha^2} \cap \mathfrak{g}_{\alpha\beta} \cap \mathfrak{g}_{-\beta^2}.$$

If $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$ then all three spaces coincide, so that $-\beta^2 = \alpha\beta = -\alpha^2$, i.e., $\alpha + \beta = 0$. \square

Definition 2.13. A CPA-structure on \mathfrak{g} is called *nil-inner*, if it can be written as $x \cdot y = [\varphi(x), y]$ with a nilpotent Lie algebra homomorphism $\varphi \in \text{End}(\mathfrak{g})$.

The trivial CPA-structure is an example of a nil-inner structure. We can now obtain a general decomposition of complex inner CPA-structures.

Theorem 2.14. *Let \mathfrak{g} be a complex Lie algebra and suppose that it admits an inner CPA-structure with $\varphi \in \text{End}(\mathfrak{g})$. Then \mathfrak{g} decomposes into the sum of φ -invariant ideals*

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$$

with the following properties:

- (1) $\varphi|_{\mathfrak{n}}$ is a nilpotent endomorphism of \mathfrak{n} such that the CPA-structure on \mathfrak{n} is nil-inner.
- (2) $\varphi|_{\mathfrak{h}}$ is an automorphism of \mathfrak{h} , and we have $[[\mathfrak{h}, \mathfrak{h}], [\mathfrak{h}, \mathfrak{h}]] = 0$.

Proof. Consider the eigenspace decomposition

$$\mathfrak{g} = \bigoplus_{\alpha} \mathfrak{g}_{\alpha} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_{\alpha}$$

of \mathfrak{g} with respect to the Lie algebra homomorphism φ , with $\mathfrak{n} = \mathfrak{g}_0$ and $\mathfrak{h} = \bigoplus_{\alpha \neq 0} \mathfrak{g}_{\alpha}$. Both \mathfrak{n} and \mathfrak{h} are Lie ideals, and hence ideals by Lemma 2.11, since $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] \subseteq \mathfrak{g}_{-\alpha^2}$ implies that $[\mathfrak{n}, \mathfrak{g}] \subseteq \mathfrak{n}$ and $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$. Moreover, both \mathfrak{n} and \mathfrak{h} are invariant under φ , so that the restrictions of φ to \mathfrak{n} and \mathfrak{h} are well-defined. Clearly the restriction of φ to \mathfrak{n} is nilpotent, and since all generalized eigenvalues of \mathfrak{h} are nonzero, the restriction of φ to \mathfrak{h} is an automorphism. It remains to show that \mathfrak{h} is metabelian, i.e., to show that

$$[[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}], [\mathfrak{g}_{\gamma}, \mathfrak{g}_{\delta}]] = 0$$

for all $\alpha, \beta, \gamma, \delta \neq 0$. Suppose this is not the case. Then Lemma 2.12 yields

$$\begin{aligned} \alpha + \beta &= 0, \\ \gamma + \delta &= 0, \\ \alpha\beta + \gamma\delta &= 0. \end{aligned}$$

Setting $\beta = -\alpha$, $\gamma = \alpha i$ and $\delta = -\alpha i$ the bracket takes the form $[[\mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha}], [\mathfrak{g}_{\alpha i}, \mathfrak{g}_{-\alpha i}]] \neq 0$. We may apply the Jacobi identity here in two ways:

$$[[\mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha}], [\mathfrak{g}_{\alpha i}, \mathfrak{g}_{-\alpha i}]] \subseteq [\mathfrak{g}_{\alpha i}, [\mathfrak{g}_{-\alpha i}, [\mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha}]]] + [\mathfrak{g}_{-\alpha i}, [[\mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha}], \mathfrak{g}_{\alpha i}]],$$

and

$$[[\mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha}], [\mathfrak{g}_{\alpha i}, \mathfrak{g}_{-\alpha i}]] \subseteq [[\mathfrak{g}_{-\alpha}, [\mathfrak{g}_{\alpha i}, \mathfrak{g}_{-\alpha i}]], \mathfrak{g}_{\alpha}] + [[[\mathfrak{g}_{\alpha i}, \mathfrak{g}_{-\alpha i}], \mathfrak{g}_{\alpha}], \mathfrak{g}_{-\alpha}].$$

In each case, at least one of the terms on the right hand side must be nonzero. The first case gives us that either $0 \neq [\mathfrak{g}_{-\alpha i}, [\mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha}]] \subseteq [\mathfrak{g}_{-\alpha i}, \mathfrak{g}_{-\alpha^2}]$, i.e., that $-\alpha i - \alpha^2 = 0$, or $0 \neq [\mathfrak{g}_{\alpha i}, [\mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha}]] \subseteq [\mathfrak{g}_{\alpha i}, \mathfrak{g}_{-\alpha^2}]$, i.e., that $\alpha i - \alpha^2 = 0$. This means $\alpha = \pm i$. The second case gives us that either $0 \neq [\mathfrak{g}_{-\alpha}, [\mathfrak{g}_{\alpha i}, \mathfrak{g}_{-\alpha i}]] \subseteq [\mathfrak{g}_{-\alpha}, \mathfrak{g}_{\alpha^2}]$, i.e., that $\alpha^2 - \alpha = 0$, or $0 \neq [\mathfrak{g}_{\alpha}, [\mathfrak{g}_{\alpha i}, \mathfrak{g}_{-\alpha i}]] \subseteq [\mathfrak{g}_{\alpha}, \mathfrak{g}_{\alpha^2}]$, i.e.,

that $\alpha^2 + \alpha = 0$. This means $\alpha = \pm 1$. So we must have both $\alpha = \pm i$ and $\alpha = \pm 1$, which is impossible. \square

Corollary 2.15. *Let \mathfrak{g} be a Lie algebra over K admitting a non-degenerate inner CPA-structure. Then \mathfrak{g} is metabelian.*

Proof. Complexifying \mathfrak{g} the above Theorem implies that $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ and \mathfrak{h} is metabelian. Since $\ker(\varphi) \subseteq \ker(L) = 0$ we have $\mathfrak{n} = 0$ and $\mathfrak{g} = \mathfrak{h}$. Now \mathfrak{g} is metabelian over \mathbb{C} if and only if \mathfrak{g} is metabelian over K . \square

Let \mathfrak{b} be the standard Borel subalgebra of $\mathfrak{sl}_2(K)$ with basis $e_1 = E_{12}$, $e_2 = E_{11} - E_{22}$ and Lie bracket $[e_1, e_2] = -2e_1$. Here E_{ij} denotes the matrix with entry 1 at position (i, j) , and entries 0 otherwise.

Example 2.16. *Every CPA-structure on the Borel subalgebra \mathfrak{b} of $\mathfrak{sl}_2(K)$ is inner, and is of the form*

$$L(e_1) = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \quad L(e_2) = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}$$

for $\alpha, \beta \in K$ such that $\alpha(\alpha - 2) = 0$.

Indeed, since \mathfrak{b} is complete, every CPA-structure on \mathfrak{b} is inner by Corollary 2.9. A short computation shows that we have $L(x) = \text{ad}(\varphi(x))$ with

$$\varphi = \frac{1}{2} \begin{pmatrix} -\alpha & -\beta \\ 0 & \alpha \end{pmatrix}$$

and $\alpha(\alpha - 2) = 0$. Note that $\varphi^2 = 0$ for $\alpha = 0$, and $\varphi^2 = I$ for $\alpha = 2$. The latter structure is non-degenerate, so that \mathfrak{b} is metabelian according to Corollary 2.15. Of course, this is obvious anyway.

3. CPA-STRUCTURES ON PERFECT LIE ALGEBRAS

For this section we will assume that all Lie algebras are complex. We start with CPA-structures on semisimple Lie algebras, where we give another proof of Proposition 5.4 and Corollary 5.5 in [10], without using the structure results of [13]:

Proposition 3.1. *Any CPA-structure on a semisimple Lie algebra is trivial. Furthermore any CPA-structure on a Lie algebra \mathfrak{g} satisfies $\mathfrak{g} \cdot \mathfrak{g} \subseteq \text{rad}(\mathfrak{g})$.*

Proof. Let (A, \cdot) be a CPA-structure on a semisimple Lie algebra \mathfrak{s} . Then it is inner by Corollary 2.9, i.e., given by $L(x) = \text{ad}(\varphi(x))$. We have $I_\infty^{[k]} \cdot \mathfrak{s} = 0$ for the ideal I_∞ of Proposition 2.6. Since I_∞ is invariant by Lemma 2.11 the quotient CPA-structure on \mathfrak{s}/I_∞ is also inner, and nondegenerate. Theorem 2.14 implies that the Lie algebra \mathfrak{s}/I_∞ is metabelian, hence solvable. Since \mathfrak{s} is perfect, any solvable quotient is trivial. Hence we have $\mathfrak{s} = I_\infty$ and $0 = I_\infty^{[k]} \cdot \mathfrak{s} = \mathfrak{s}^{[k]} \cdot \mathfrak{s} = \mathfrak{s} \cdot \mathfrak{s}$. Hence the CPA-structure on \mathfrak{s} is trivial. The second part follows by considering the semisimple quotient $\mathfrak{g}/\text{rad}(\mathfrak{g})$. \square

Lemma 3.2. *Let \mathfrak{s} be a semisimple Lie algebra. Then there exist Lie algebra generators $\{s_i \mid 1 \leq i \leq m\}$ of \mathfrak{s} such that for every linear representation $\psi : \mathfrak{s} \rightarrow \mathfrak{gl}(V)$, all $\psi(s_i)$ are nilpotent.*

Proof. Let $\{e_i, f_i, h_i \mid 1 \leq i \leq k\}$ be the Chevalley-Serre generators for \mathfrak{s} . Each triple (e_i, f_i, h_i) generates a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, and ψ restricted to it is a representation. By the classification of representations of $\mathfrak{sl}_2(\mathbb{C})$ we know that $\psi(e_i)$ and $\psi(f_i)$ are nilpotent. It follows that $\{e_i, f_i \mid 1 \leq i \leq k\}$ is a set of generators for \mathfrak{s} such that all $\psi(e_i)$ and all $\psi(f_i)$ are nilpotent. \square

We are now able to prove Conjecture 5.21 of [10].

Theorem 3.3. *Any CPA-structure on a perfect Lie algebra \mathfrak{g} is trivial, i.e., satisfies $\mathfrak{g} \cdot \mathfrak{g} = 0$.*

Proof. Let \mathfrak{g} be a perfect Lie algebra with Levi subalgebra \mathfrak{s} and solvable radical $\text{rad}(\mathfrak{g}) = \mathfrak{a}$. We have $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{a}$. Denote by $\text{Der}(\mathfrak{g}, \mathfrak{a})$ the space of those derivations $D \in \text{Der}(\mathfrak{g})$ satisfying $D(\mathfrak{g}) \subseteq \mathfrak{a}$. For the proof it is sufficient to show that $\mathfrak{s} \cdot \mathfrak{g} = 0$, since \mathfrak{g} is perfect and hence \mathfrak{s} generates \mathfrak{g} as a Lie ideal by Lemma 5.15 in [10]. By Corollary 5.17 in [10] we may assume that \mathfrak{a} is *abelian*. Decompose \mathfrak{a} into irreducible \mathfrak{s} -modules $\mathfrak{a} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_m$. By Proposition 3.1 we have $\mathfrak{g} \cdot \mathfrak{g} \subseteq \mathfrak{a}$, i.e., $L(\mathfrak{g})(\mathfrak{g}) \subseteq \mathfrak{a}$, and hence $L(\mathfrak{g}) \subseteq \text{Der}(\mathfrak{g}, \mathfrak{a})$. Lemma 5.18 in [10] gives a natural splitting

$$\text{Der}(\mathfrak{g}, \mathfrak{a}) = \text{Der}_{\mathfrak{s}}(\mathfrak{a}) \ltimes Z^1(\mathfrak{s}, \mathfrak{a}),$$

where

$$\text{Der}_{\mathfrak{s}}(\mathfrak{a}) = \{d \in \text{Der}(\mathfrak{a}) \mid \varphi(x)d(a) = d(\varphi(x)a) \ \forall x \in \mathfrak{s}, a \in \mathfrak{a}\}$$

with $L(x) = \text{ad}(\varphi(x))$. Since \mathfrak{s} is semisimple, Whitehead's first Lemma implies that

$$\begin{aligned} \mathfrak{s} \cdot \mathfrak{g} &= \mathfrak{g} \cdot \mathfrak{s} \\ &= Z^1(\mathfrak{s}, \mathfrak{a})(\mathfrak{g}) \\ &= B^1(\mathfrak{s}, \mathfrak{a})(\mathfrak{s}) \\ &= [\mathfrak{s}, \mathfrak{a}] \\ &= [\mathfrak{s}, \mathfrak{a}_1] + \cdots + [\mathfrak{s}, \mathfrak{a}_m] \end{aligned}$$

for all $s \in \mathfrak{s}$. On the other hand, we have the natural embeddings of vector spaces

$$\text{Der}_{\mathfrak{s}}(\mathfrak{a}) \subseteq \text{Hom}_{\mathfrak{s}}(\mathfrak{a}) \subseteq \bigoplus_{i,j} \text{Hom}(\mathfrak{a}_i, \mathfrak{a}_j).$$

Hence for every $s \in \mathfrak{s}$ there exist linear maps $f_{j,i}^s \in \text{Hom}_{\mathfrak{s}}(\mathfrak{a}_i, \mathfrak{a}_j)$ such that

$$s \cdot v_i = \sum_{k=1}^m f_{k,i}^s(v_i)$$

for all $v_i \in \mathfrak{a}_i$, for every i . Altogether we obtain $f_{j,i}^s(\mathfrak{a}_i) \subseteq [\mathfrak{s}, \mathfrak{a}_j]$ for all $j, i \in \{1, \dots, m\}$.

Suppose that $s \in \mathfrak{s}$ is an element such that $[\mathfrak{s}, \mathfrak{a}_j] \subsetneq \mathfrak{a}_j$ for all j . Then Schur's Lemma applied to the simple \mathfrak{s} -modules \mathfrak{a}_j implies that $f_{j,i}^s = 0$ for all i, j , so that $s \cdot \mathfrak{a} = 0$. Now Lemma 3.2 applied to the linear representations $\psi_j = \text{ad}_{\mathfrak{a}_j}$ gives us a set of generators $\{s_1, \dots, s_k\}$ of \mathfrak{s} such that $\text{im}(\psi_j(s_i)) = [\mathfrak{s}_i, \mathfrak{a}_j] \subsetneq \mathfrak{a}_j$ for all i, j , since all $\psi_j(s_i)$ are nilpotent. Thus we have $s_i \cdot \mathfrak{a} = 0$ for all i . Since the s_i generate \mathfrak{s} this means that $\mathfrak{s} \cdot \mathfrak{a} = 0$, and hence $L(\mathfrak{s}) \subseteq Z^1(\mathfrak{s}, \mathfrak{a})$. By Lemma 5.18 in [10] $Z^1(\mathfrak{s}, \mathfrak{a})$ is abelian, so that $L(\mathfrak{s})$ is both abelian and semisimple, hence trivial. We obtain $L(\mathfrak{s}) = 0$, so that $\mathfrak{s} \cdot \mathfrak{g} = 0$ and the proof is finished. \square

We can generalize the last result as follows.

Theorem 3.4. *Let \mathfrak{p} be a perfect subalgebra of a Lie algebra \mathfrak{g} . Then every CPA-structure on \mathfrak{g} satisfies $\mathfrak{p} \cdot \mathfrak{g} = 0$.*

Proof. Let \mathfrak{t} be a Levi complement of \mathfrak{p} . Then $\mathfrak{p} \cdot \mathfrak{g} = 0$ if and only if $\mathfrak{t} \cdot \mathfrak{g} = 0$, again by Lemma 5.15 in [10] and the fact that for a set $X \subseteq \ker(L)$ the ideal in \mathfrak{g} generated by X also lies in $\ker(L)$. We have $\mathfrak{t} \cdot \mathfrak{g} \subseteq \mathfrak{s} \cdot \mathfrak{g}$ for some Levi complement \mathfrak{s} of \mathfrak{g} . Hence it is enough to show that $\mathfrak{s} \cdot \mathfrak{g} = 0$ for all Levi complements \mathfrak{s} of \mathfrak{g} . Suppose first that \mathfrak{g} has no proper characteristic ideal I with $0 \subsetneq I \subsetneq \text{rad}(\mathfrak{g})$. Then $\text{rad}(\mathfrak{g})$ is abelian, because otherwise $[\text{rad}(\mathfrak{g}), \text{rad}(\mathfrak{g})]$ would be a proper characteristic ideal. Furthermore \mathfrak{g} is of the form $\mathfrak{g} = \mathfrak{s} \ltimes V^n$ with an irreducible \mathfrak{s} -module. If V is the trivial module, then \mathfrak{g} is reductive and we have $\mathfrak{s} \cdot \mathfrak{g} = 0$ by Corollary 5.6 of [10]. Otherwise $\mathfrak{g} = \mathfrak{s} \ltimes V^n$ is perfect, and $\mathfrak{s} \cdot \mathfrak{g} = 0$ by Theorem 3.3.

It remains to study the case where \mathfrak{g} admits a proper characteristic ideal $0 \subsetneq I \subsetneq \text{rad}(\mathfrak{g})$. Either we have $\mathfrak{s} \cdot \mathfrak{g} = 0$ and we are done, or there exists a Lie algebra \mathfrak{g} with $\mathfrak{s} \cdot \mathfrak{g} \neq 0$. We may choose \mathfrak{g} so that it is of minimal dimension. By Proposition 3.1 we have $\mathfrak{s} \cdot \mathfrak{g} \subseteq \text{rad}(\mathfrak{g})$, so that $\text{rad}(\mathfrak{g}) \neq 0$. Since \mathfrak{s} is semisimple, the \mathfrak{g} -module \mathfrak{g} given by the representation $x \mapsto L(x)$ has a \mathfrak{g} -module complement U with $\mathfrak{g} = U \oplus \text{rad}(\mathfrak{g})$. Using $\mathfrak{s} \cdot \mathfrak{g} \subseteq \text{rad}(\mathfrak{g})$ we obtain $\mathfrak{s} \cdot U = 0$. Since I is invariant under the \mathfrak{s} -action, we have a module complement K with $\text{rad}(\mathfrak{g}) = K \oplus I$. The quotient algebra \mathfrak{g}/I then is isomorphic to $\mathfrak{s} \ltimes K/I$, and the minimality of \mathfrak{g} implies $\mathfrak{s} \cdot \mathfrak{g} \subseteq I$, so that $\mathfrak{s} \cdot K \subseteq K \cap I = 0$. We see that the Lie algebra $\mathfrak{s} \ltimes I$ is closed under the CPA-product: since I is a characteristic ideal of \mathfrak{g} we have $\mathfrak{g} \cdot I \subseteq I$, and

$$(\mathfrak{s} \ltimes I) \cdot (\mathfrak{s} \ltimes I) \subseteq \mathfrak{s} \cdot \mathfrak{g} + \mathfrak{g} \cdot I \subseteq \mathfrak{s} \ltimes I.$$

Since \mathfrak{g} is minimal it follows that $\mathfrak{s} \cdot I = 0$, and

$$\mathfrak{s} \cdot \mathfrak{g} = \mathfrak{s} \cdot (U + K + I) = \mathfrak{s} \cdot U + \mathfrak{s} \cdot K + \mathfrak{s} \cdot I = 0.$$

This is a contradiction, and the proof is finished. \square

Corollary 3.5. *Suppose that \mathfrak{g} admits a nondegenerate CPA-product. Then \mathfrak{g} is solvable.*

Proof. Let \mathfrak{s} be a Levi subalgebra of \mathfrak{g} . Then $\mathfrak{s} \cdot \mathfrak{g} = 0$ by Theorem 3.4, so that $\mathfrak{s} \subseteq \ker(L) = 0$. Hence $\text{rad}(\mathfrak{g}) = \mathfrak{g}$, and \mathfrak{g} is solvable. \square

Since we know that a perfect Lie algebra only admits the trivial CPA-structure, it is natural to ask for the converse. Given a non-perfect Lie algebra \mathfrak{g} . Can we construct a non-trivial CPA-structures on \mathfrak{g} ? The following example shows that this is not always possible.

Example 3.6. *Let \mathfrak{g} denote the Lie subalgebra of $\mathfrak{sl}_3(\mathbb{C})$ of dimension 6 with basis*

$$(e_1, \dots, e_6) = (E_{12}, E_{13}, E_{21}, E_{23}, E_{11} - E_{22}, E_{22} - E_{33}).$$

Then \mathfrak{g} is not perfect and admits only the trivial CPA-product.

The Lie brackets are given by

$$\begin{aligned} [e_1, e_3] &= e_5, [e_1, e_4] = e_2, [e_1, e_5] = -2e_1, [e_1, e_6] = e_1, \\ [e_2, e_3] &= -e_4, [e_2, e_5] = -e_2, [e_2, e_6] = -e_2, [e_3, e_5] = 2e_3, \\ [e_3, e_6] &= -e_3, [e_4, e_5] = e_4, [e_4, e_6] = -2e_4. \end{aligned}$$

We have $\dim[\mathfrak{g}, \mathfrak{g}] = 5$, so that \mathfrak{g} is not perfect. For a given CPA-structure we know by Theorem 3.4 that $\mathfrak{p} \cdot \mathfrak{g} = 0$ for the perfect subalgebra $\mathfrak{p} = \text{span}\{e_1, \dots, e_5\}$. It is now easy to see that the CPA-product on \mathfrak{g} is trivial.

On the other hand we will show that every solvable Lie algebra \mathfrak{g} admits a non-trivial CPA-structure. Here we distinguish two cases, namely whether or not \mathfrak{g} has trivial center.

Proposition 3.7. *Let \mathfrak{g} be a solvable Lie algebra with trivial center. Then \mathfrak{g} admits a non-trivial nil-inner CPA-structure.*

Proof. By Lie's theorem there exists a nonzero common eigenvector $v \in \mathfrak{g}$ and a linear functional $\lambda: \mathfrak{g} \rightarrow \mathbb{C}$ such that $[x, v] = \lambda(x)v$ for all $x \in \mathfrak{g}$. We have

$$\begin{aligned} \lambda([x, y])v &= [[x, y], v] \\ &= [x, [y, v]] - [y, [x, v]] \\ &= (\lambda(x)\lambda(y) - \lambda(y)\lambda(x))v \\ &= 0. \end{aligned}$$

Hence $x \cdot_v y := [x, [y, v]] = \lambda(x)\lambda(y)v$ defines a CPA-structure on \mathfrak{g} . It is non-trivial, because otherwise the center of \mathfrak{g} were non-trivial. \square

Proposition 3.8. *Let \mathfrak{g} be a non-perfect Lie algebra with non-trivial center. Then \mathfrak{g} admits a non-trivial CPA-product.*

Proof. Suppose first that $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}] \neq 0$, and select a nonzero z from it. Since \mathfrak{g} is not perfect we may choose a 1-codimensional ideal I of \mathfrak{g} with $I \supseteq [\mathfrak{g}, \mathfrak{g}]$. Fix a basis (e_2, \dots, e_n) for I and a generator e_1 for the vector space complement of I in \mathfrak{g} . Then \mathfrak{g} is a semidirect product $\mathbb{C}e_1 \ltimes I$. Using the nonzero $z \in Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ define a non-trivial CPA-structure on \mathfrak{g} by

$$\left(\sum_{i=1}^n \alpha_i e_i \right) \cdot \left(\sum_{i=1}^n \beta_i e_i \right) := \alpha_1 \beta_1 z.$$

Now assume that $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}] = 0$. Then \mathfrak{g} admits an abelian factor, because $Z(\mathfrak{g}) \neq 0$. So we can write $\mathfrak{g} = \mathbb{C}e_1 \oplus \mathfrak{h}$ for some ideal \mathfrak{h} in \mathfrak{g} . Let (e_2, \dots, e_n) be a basis of \mathfrak{h} and define a non-trivial CPA-structure on \mathfrak{g} as before but replacing z by e_1 on the RHS. Note that in both cases the CPA-product is even associative. \square

Corollary 3.9. *Let \mathfrak{g} be a non-trivial solvable Lie algebra. Then \mathfrak{g} admits a non-trivial CPA-structure.*

4. CPA-STRUCTURES ON PARABOLIC SUBALGEBRAS OF SEMSIMPLE LIE ALGEBRAS

For this section we will assume that all Lie algebras are complex. The following construction yields a class of CPA-structures which is important for the case of parabolic subalgebras of semisimple Lie algebras.

Proposition 4.1. *Let I be an ideal in \mathfrak{g} with center $\mathfrak{z} = Z(I)$ such that \mathfrak{g}/I is abelian. Then every 1-cocycle $f \in Z^1(\mathfrak{g}/I, \mathfrak{z})$ defines an associative nil-inner CPA-structure on \mathfrak{g} by*

$$x \cdot y = [f(\bar{x}), y]$$

for all $x, y \in \mathfrak{g}$.

Proof. Note that \mathfrak{z} is a characteristic ideal of I , and hence an ideal of \mathfrak{g} . Therefore \mathfrak{g} acts on \mathfrak{z} by the adjoint action $x \circ z = [x, z]$ for all $x \in \mathfrak{g}$ and $z \in \mathfrak{z}$. Since I acts trivially on \mathfrak{z} we obtain

an induced action on the quotient \mathfrak{g}/I on \mathfrak{z} by $\bar{x} \circ z = [x, z]$. Now $Z^1(\mathfrak{g}/I, \mathfrak{z})$ consists of linear maps $f: \mathfrak{g}/I \rightarrow \mathfrak{z}$ satisfying

$$f([\bar{x}, \bar{y}]) = -\bar{y} \circ f(\bar{x}) + \bar{x} \circ f(\bar{y})$$

Since \mathfrak{g}/I is abelian, the condition reduces to $[f(\bar{x}), y] = [f(\bar{y}), x]$ for all $x, y \in \mathfrak{g}$. We claim that $x \cdot y = [f(\bar{x}), y]$ satisfies the axioms (4), (5), (6), of a CPA-structure. By the last remark we have $x \cdot y = y \cdot x$, so that (4) is satisfied. All products $x \cdot (y \cdot z) = [f(\bar{x}), [f(\bar{y}), z]] \subseteq [\mathfrak{z}, \mathfrak{z}] = 0$ are zero, so that the CPA-product is nil-inner and associative. Furthermore we have $[x, y] \cdot z = [f([\bar{x}, \bar{y}]), z] = 0$, and hence (5) is satisfied. Finally the Jacobi identity for the bracket on I implies that

$$\begin{aligned} x \cdot [y, z] &= [f(\bar{x}), [y, z]] \\ &= [[f(\bar{x}), y], z] + [y, [f(\bar{x}), z]] \\ &= [x \cdot y, z] + [y, x \cdot z]. \end{aligned}$$

Hence also (6) is satisfied. \square

Remark 4.2. Proposition 4.1 once more implies that every non-trivial solvable Lie algebra \mathfrak{g} with trivial center admits a non-trivial CPA-structure. In fact, take $I = [\mathfrak{g}, \mathfrak{g}]$, so that the quotient \mathfrak{g}/I is abelian. By assumption $I \neq 0$, and I is nilpotent, so that $\mathfrak{z} := Z(I)$ is non-trivial. Since \mathfrak{g} has trivial center we have $[[\mathfrak{z}, \mathfrak{g}], \mathfrak{g}] \neq 0$, so that $x \cdot y := [[z, x], y]$ defines a non-trivial CPA-product for any $z \neq 0$ in \mathfrak{z} .

Definition 4.3. Denote by $\text{fix}(\mathfrak{g})$ the Lie ideal of \mathfrak{g} generated by the set

$$\{x \in \mathfrak{g} \mid \text{ad}(y)x = x \text{ for some } y \in \mathfrak{g}\}.$$

We have $\text{fix}(\mathfrak{g}) = 0$ if and only if \mathfrak{g} is nilpotent by Engel's theorem. For the other extreme we have the following result:

Lemma 4.4. *We have $\text{fix}(\mathfrak{g}) = \mathfrak{g}$ if and only if \mathfrak{g} is perfect.*

Proof. Since a perfect Lie algebra \mathfrak{g} is generated by any of its Levi subalgebras, we may assume that \mathfrak{g} is semisimple. Let $\{e_i, f_i, h_i \mid 1 \leq i \leq k\}$ be the Chevalley-Serre generators of \mathfrak{g} . We have $[\frac{1}{2}h_i, e_i] = e_i$ and $[-\frac{1}{2}h_i, f_i] = f_i$ for all i , so that $e_i, f_i \in \text{fix}(\mathfrak{g})$ for all i . Hence we also have $h_i = [e_i, f_i] \in \text{fix}(\mathfrak{g})$, so that $\mathfrak{g} \subseteq \text{fix}(\mathfrak{g}) \subseteq \mathfrak{g}$. Conversely, if $\text{fix}(\mathfrak{g}) = \mathfrak{g}$, then $\text{fix}(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]$ implies that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. \square

Lemma 4.5. *Let \mathfrak{p} be a parabolic subalgebra of a semisimple Lie algebra \mathfrak{g} , and \mathfrak{s} be a Levi subalgebra of \mathfrak{p} . Then $\text{fix}(\mathfrak{p}) = [\mathfrak{p}, \mathfrak{p}] = \mathfrak{s} \ltimes \text{nil}(\mathfrak{p})$.*

Proof. Let \mathfrak{b} be a standard Borel subalgebra of \mathfrak{g} with standard generators $h_i, x_i, i = 1, \dots, k$. Since $[\frac{1}{2}h_i, x_i] = x_i$ we obtain $x_i \in \text{fix}(\mathfrak{p})$ for all i . We have $\text{nil}(\mathfrak{b}) \subseteq \text{fix}(\mathfrak{b})$, since $\text{nil}(\mathfrak{b})$ is generated by the x_i . On the other hand, $\text{fix}(\mathfrak{b}) \subseteq [\mathfrak{b}, \mathfrak{b}] \subseteq \text{nil}(\mathfrak{b})$ yields $\text{nil}(\mathfrak{b}) = \text{fix}(\mathfrak{b})$. Furthermore we have that $\text{nil}(\mathfrak{p}) \subseteq \text{nil}(\mathfrak{b}) = \text{fix}(\mathfrak{b}) \subseteq \text{fix}(\mathfrak{p})$. By Lemma 4.4 we have $\mathfrak{s} \subseteq \text{fix}(\mathfrak{p})$, so that $\mathfrak{s} \ltimes \text{nil}(\mathfrak{p}) \subseteq \text{fix}(\mathfrak{p})$. Conversely we have $\text{fix}(\mathfrak{p}) \subseteq [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{s} \ltimes \text{nil}(\mathfrak{p})$. It is well-known that $\mathfrak{s} \ltimes \text{nil}(\mathfrak{p}) = [\mathfrak{p}, \mathfrak{p}]$ for parabolic subalgebras of semisimple Lie algebras. \square

Lemma 4.6. *Let $x \cdot y = [\varphi(x), y]$ be a nil-inner CPA-structure on \mathfrak{g} . Then $\varphi(\mathfrak{g}) \subseteq \text{nil}(\mathfrak{g})$ and $\text{fix}(\mathfrak{g}) \subseteq \ker(\varphi)$.*

Proof. We already have seen that $x \cdot y = y \cdot x$ is equivalent to the identity $[\varphi(x), y] = -[x, \varphi(y)]$. This yields $\text{ad}(\varphi(x))(y) = -\text{ad}(x)^m(\varphi^{2^m-1}(y))$ by induction on m . Since φ is nilpotent, this implies $\varphi(\mathfrak{g}) \subseteq \text{nil}(\mathfrak{g})$. Now let $x, y \in \mathfrak{g}$ with $[y, x] = x$. Then we have

$$0 = \text{ad}(\varphi(y))^m(\varphi(x)) = \varphi(\text{ad}(y)^m(x)) = \varphi(x)$$

for sufficiently large m . Since φ is a homomorphism, it also vanishes on the Lie ideal generated by such x . This means $\varphi(\text{fix}(\mathfrak{g})) = 0$. \square

We can now give a description of CPA-structures on parabolic subalgebras \mathfrak{p} of simple Lie algebras. There are two cases, namely that the Borel subalgebra of \mathfrak{p} is metabelian, or not. The metabelian case is as follows.

Lemma 4.7. *Let \mathfrak{s} be a simple Lie algebra and \mathfrak{p} a parabolic subalgebra of \mathfrak{s} . Then \mathfrak{p} is metabelian if and only if \mathfrak{s} is of type A_1 and \mathfrak{p} a Borel subalgebra.*

Proof. Suppose that \mathfrak{p} is a Borel subalgebra of A_1 . Then \mathfrak{p} is metabelian. Conversely suppose that \mathfrak{p} is metabelian. Hence \mathfrak{p} is a solvable parabolic subalgebra of \mathfrak{s} , hence a Borel subalgebra, which we denote by \mathfrak{b} now. Denote by \mathfrak{n} the nilradical of \mathfrak{b} . Since $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}$ it is enough to show that \mathfrak{n} is abelian if and only if \mathfrak{s} is of type A_1 . However we have $\dim Z(\mathfrak{n}) = 1$ for all simple Lie algebras \mathfrak{s} , see [16] section 4, so that \mathfrak{n} is abelian if and only if \mathfrak{n} is 1-dimensional. This is true if and only if \mathfrak{s} is of type A_1 , see table 2 in [16], which gives the dimensions of the nilradicals of \mathfrak{b} for all simple Lie algebras. \square

The remaining case, where \mathfrak{p} is not metabelian, is as follows.

Theorem 4.8. *Let \mathfrak{p} be a parabolic subalgebra of a simple Lie algebra \mathfrak{g} , and suppose that \mathfrak{p} is not metabelian. Denote by \mathfrak{z} the center of the ideal $I = [\mathfrak{p}, \mathfrak{p}]$. Then there is a bijective correspondence between CPA-products on \mathfrak{p} and elements $z \in \mathfrak{z}$, given by*

$$x \cdot y = [[z, x], y].$$

Proof. Since parabolic subalgebras of semisimple Lie algebras are complete, any CPA-structure on \mathfrak{p} is inner by Corollary 2.9. In fact, any CPA-structure on \mathfrak{p} is nil-inner by Theorem 2.14, since \mathfrak{p} is indecomposable and not metabelian. Writing $x \cdot y = [\varphi(x), y]$ we have $\varphi(I) \subseteq \varphi(\text{fix}(\mathfrak{p})) = 0$ by Lemma 4.5 and Lemma 4.6. The identity $x \cdot y = y \cdot x$ yields $[\varphi(\mathfrak{p}), I] = [\varphi(I), \mathfrak{p}] = 0$. Lemma 4.6 gives $\varphi(\mathfrak{p}) \subseteq \text{nil}(\mathfrak{p}) \subseteq I$, so that $\varphi(\mathfrak{p}) \subseteq \mathfrak{z}$. Hence φ may be identified with its restriction $\varphi: \mathfrak{p} \rightarrow \mathfrak{z}$. By Lemma 4.6 we obtain $I \subseteq \ker(\varphi)$, so that φ projects to a quotient map $f: \mathfrak{p}/I \rightarrow \mathfrak{z}$, $\bar{x} \mapsto \varphi(x)$. By commutativity of the product we obtain $[f(\bar{x}), y] = [f(\bar{y}), x]$ for all $x, y \in \mathfrak{p}$. Since \mathfrak{p}/I is abelian and $f(\mathfrak{p}) \subseteq \mathfrak{z}$ this implies $f \in Z^1(\mathfrak{p}/I, \mathfrak{z})$ and $x \cdot y = [f(\bar{x}), y]$. Conversely, every 1-cocycle $f \in Z^1(\mathfrak{p}/I, \mathfrak{z})$ defines a nil-inner CPA product on \mathfrak{p} by Proposition 4.1. Since \mathfrak{p}/I is abelian and $Z(\mathfrak{p}) = 0$ we have $H^0(\mathfrak{p}/I, \mathfrak{z}) \subseteq Z(\mathfrak{p}) = 0$. This implies $H^n(\mathfrak{p}/I, \mathfrak{z}) = 0$ for all $n \geq 0$, and in particular for $n = 1$ we obtain $x \cdot y = [[z, x], y]$ for $z \in \mathfrak{z}$. Hence we have a bijective correspondence between CPA-products on \mathfrak{p} and elements $z \in \mathfrak{z}$. \square

We can now review Example 3.6.

Example 4.9. *The 6-dimensional parabolic subalgebra \mathfrak{g} of $\mathfrak{sl}_3(\mathbb{C})$ given in Example 3.6 admits only the trivial CPA-product.*

With the notations of Theorem 4.8 we have $\mathfrak{s} = \langle e_1, e_3, e_5 \rangle$ acting on $\text{nil}(\mathfrak{g}) = \langle e_2, e_4 \rangle$ by the irreducible action of dimension 2. In particular we have $Z(I) = Z(\mathfrak{s} \ltimes \text{nil}(\mathfrak{g})) = 0$, so that all CPA-products on \mathfrak{g} vanish.

We can now describe explicitly all CPA-products on parabolic subalgebras of simple Lie algebras \mathfrak{s} . We may assume that \mathfrak{s} has rank at least 2. For the rank one case see Example 2.16. We demonstrate the result for standard Borel subalgebras \mathfrak{b} of simple Lie algebras type A_n . Let h_i, x_i , $i = 1, \dots, k$ be the standard generators of \mathfrak{b} , and let z be a generator of $Z(\text{nil}(\mathfrak{b}))$.

Proposition 4.10. *Suppose that $\mathfrak{s} = \mathfrak{sl}_{k+1}(\mathbb{C})$ with $k \geq 2$, and \mathfrak{b} a standard Borel subalgebra of \mathfrak{s} . Then we have $[h_1, z] = [h_k, z] = z$ and $[h_i, z] = 0$ for all $i \neq 1, k$. All CPA-products on \mathfrak{b} are scalar multiples of the following product*

$$h_1 \cdot h_1 = h_k \cdot h_k = h_1 \cdot h_k = h_k \cdot h_1 = z.$$

We would like to extend Theorem 4.8 to parabolic subalgebras of semisimple Lie algebras \mathfrak{s} . Since parabolic subalgebras of semisimple Lie algebras are complete, we first study the case of complete Lie algebras. The following definition is given in [15].

Definition 4.11. A complete Lie algebra \mathfrak{g} is called *simply-complete*, if no non-trivial ideal in \mathfrak{g} is complete.

Meng [15] showed that every complex complete Lie algebra \mathfrak{g} decomposes into a direct sum of simply-complete ideals, and this decomposition is unique up to permutation of the ideals.

Proposition 4.12. *Let $\mathfrak{q}_1, \dots, \mathfrak{q}_n$ be simply-complete Lie algebras, each with a CPA-product. Then the direct Lie algebra sum admits a CPA-product, which is given componentwise:*

$$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = (x_1 \cdot y_1, \dots, x_n \cdot y_n).$$

Conversely, for any complete Lie algebra $\mathfrak{q} = \mathfrak{q}_1 \oplus \dots \oplus \mathfrak{q}_n$ with simply-complete ideals \mathfrak{q}_i , any CPA-product on \mathfrak{q} is given as above.

Proof. The first part is clear. For the second part we need only show that $\mathfrak{q}_i \cdot \mathfrak{q}_j \subseteq \mathfrak{q}_i \cap \mathfrak{q}_j$. Because all derivations of \mathfrak{q} are inner, we have $\mathfrak{q}_i \cdot \mathfrak{q}_j \subseteq \text{Der}(\mathfrak{q})(\mathfrak{q}_j) \subseteq \mathfrak{q}_i$, and because the CPA-product is commutative also $\mathfrak{q}_i \cdot \mathfrak{q}_j \subseteq \mathfrak{q}_j$. \square

We think that this will lead to a classification of CPA-products on parabolic subalgebras of semisimple Lie algebras. Furthermore one might also wish to extend the results to parabolic subalgebras of *reductive* Lie algebras. Let \mathfrak{q} be a parabolic subalgebra of a reductive Lie algebra \mathfrak{g} . Then $\text{Der}(\mathfrak{q}) = \mathfrak{L} \oplus \text{ad}(\mathfrak{q})$ as a Lie algebra direct sum, where \mathfrak{L} is the set of all linear transformations $D: \mathfrak{q} \rightarrow Z(\mathfrak{q})$ such that $D([\mathfrak{q}, \mathfrak{q}]) = 0$, see [1]. Furthermore we have $Z(\mathfrak{q}) = Z(\mathfrak{g})$. However, the situation here is more complicated than before.

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